Uniqueness of radial centers of parallel bodies

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Abstract

We show the uniqueness of the radial centers of any order α of a parallel body of a convex body Ω in $\mathbb{R}^m$ at distance δ if δ is greater than the diameter of Ω multiplied by a constant which depends only on the dimension m.

Key words and phrases. Riesz potential, renormalization, centroid.
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1 Introduction

Let Ω be a *body* in $\mathbb{R}^m$ $(m \geq 2)$, i.e. a compact set which is a closure of its interior, with a piecewise C^1 boundary. Consider a potential of the form

$$V_\Omega^{(\alpha)}(x) = \int_\Omega |x-y|^{\alpha-m}\, d\mu(y) \qquad (\alpha > 0),$$

where μ is the standard Lesbegue measure of $\mathbb{R}^m$. It is a singular integral when $\alpha < m$ and $x \in \Omega$. When $0 < \alpha < m$ it is the *Riesz potential* of (the characteristic function χ_Ω of) Ω.

In particular, when Ω is convex and $x \in \overset{\circ}{\Omega}$, $V_\Omega^{(\alpha)}(x)$ can be expressed as

$$V_\Omega^{(\alpha)}(x) = \frac{1}{\alpha}\int_{S^{m-1}} \Big(\rho_{\Omega_{-x}}(v)\Big)^\alpha\, d\sigma(v)$$

where σ is the standard Lebesgue measure of S^{m-1} and $\rho_{\Omega_{-x}} : S^{m-1} \to \mathbb{R}_{>0}$ is a *radial function* of $\Omega_{-x} = \{y - x \,|\, y \in \Omega\}$ given by $\rho_{\Omega_{-x}}(v) = \sup\{a \geq 0 \,|\, x + av \in \Omega\}$. Thus $V_\Omega^{(\alpha)}(x)$ coincides with the *dual mixed volume* as introduced by Lutwak ([L1, L2]) up to multiplication by a constant.

In [O3] we defined an $r^{\alpha-m}$*-center of* Ω. It is a point where the extreme value of $V_\Omega^{(\alpha)}$ (minimum or maximum according to the value of α) is attained when $\alpha \neq m$. (The case when $\alpha = m$ will be addressed later.) For example, the center of mass is an r^2-center. When Ω is convex, an $r^{\alpha-m}$-center $(\alpha \neq m)$ coincides with the *radial center of order* α, which was introduced in [M] for $0 < \alpha \leq 1$ and in [H] in general $(\alpha \neq 0)$. An $r^{\alpha-m}$-center of a body Ω exists for any α and is unique if $\alpha \geq m+1$ or if $\alpha \leq 1$ and Ω is convex ([O3]). It was conjectured that a convex subset Ω has a unique $r^{\alpha-m}$-center for any α.

In this paper we show the uniqueness of an $r^{\alpha-m}$-center for any α when Ω is close to a ball in some sense. To be precise, we show that there is a positive function $\varphi(m)$ such that for any convex body $\widetilde{\Omega}$ with a piecewise C^1 boundary, if $\delta \geq \varphi(m) \cdot \mathrm{diam}(\widetilde{\Omega})$ then a δ*-parallel body* of $\widetilde{\Omega}$ has a unique $r^{\alpha-m}$-center for any α. Here, a δ-parallel body of $\widetilde{\Omega}$ is the closure of a δ-tubular neighbouhood of $\widetilde{\Omega}$, and is denoted by $\widetilde{\Omega} + \delta B^m$. The proof has two steps. First we show that a center can appear only in $\widetilde{\Omega}$ by the so-called *moving plane method* in analysis ([GNN]). Then we show that $V^{(\alpha)}_{\widetilde{\Omega}+\delta B^m}$ is convex (or concave according to the value of α) on $\widetilde{\Omega}$ using the boundary integral expression of the second derivatives of $V_\Omega^{(\alpha)}$.

Throughout the paper, $\overset{\circ}{X}$, $\overline{X}$, X^c, and $\mathrm{conv}(X)$ denote the interior, the closure, the complement, and the convex hull of X respectively. We denote the standard Lesbegue measure of $\mathbb{R}^m$ by μ, and that of $\partial\Omega$ and other $(m-1)$-dimensional spaces like S^{m-1} by σ.

2 Preliminaries from [O3]

In this section we introduce some of the results of [O3] which are necessary for our study.

First remark that if we define

$$X - Y = (X \setminus (X \cap Y)) \cup -(Y \setminus (X \cap Y)) \qquad (X, Y \subset \mathbb{R}^m),$$

where the second term is equipped with the reverse orientation, then

$$V^{(\alpha)}_{\Omega_1 - \Omega_2}(x) = V^{(\alpha)}_{\Omega_1}(x) - V^{(\alpha)}_{\Omega_2}(x) \qquad (x \in \overset{\circ}{\Omega}_1 \cap \overset{\circ}{\Omega}_2)$$

for any α.

2.1 Boundary integral expression of the derivatives

The first derivatives of $V^{(\alpha)}_{\Omega}$ can be expressed by the boundary integral as

$$\frac{\partial V^{(\alpha)}_{\Omega}}{\partial x_j}(x) = -\int_{\partial\Omega} |x-y|^{\alpha-m}\, e_j \cdot n \, d\sigma(y) \tag{2.1}$$

for any j $(1 \le j \le m)$ if $x = (x_1, \dots, x_m) \notin \partial\Omega$, where n is a unit outer normal vector to $\partial\Omega$ at y, e_j is the j-th unit vector of $\mathbb{R}^m$, and σ denotes the standard Lebesgue measure of $\partial\Omega$. This is because

$$\frac{\partial\, r^{\alpha-m}}{\partial x_j} = -\frac{\partial\, r^{\alpha-m}}{\partial y_j} = -\mathrm{div}_y \left(r^{\alpha-m}\, e_j\right).$$

It follows that the second derivatives satisfy

$$\frac{\partial^2 V^{(\alpha)}_{\Omega}}{\partial {x_j}^2}(x) = -(\alpha-m)\int_{\partial\Omega} |x-y|^{\alpha-m-2}(x_j-y_j)\, e_j \cdot n \, d\sigma(y) \tag{2.2}$$

for any α if $x \notin \partial\Omega$ (or for any x if $\alpha > 2$). Furthermore, if $x \in \Omega^c$ then for any α

$$\begin{aligned}\frac{\partial^2 V^{(\alpha)}_{\Omega}}{\partial {x_j}^2}(x) &= (\alpha-m)\int_{\Omega} |x-y|^{\alpha-m-4}\left((\alpha-m-2)(x_j-y_j)^2 + |x-y|^2\right) d\mu(y) &(2.3)\\ &= (\alpha-m)\int_{\Omega} |x-y|^{\alpha-m-4}\left((\alpha-m-1)(x_j-y_j)^2 + \sum_{i\ne j}(x_i-y_i)^2\right) d\mu(y). &(2.4)\end{aligned}$$

2.2 Definition of the $r^{\alpha-m}$-centers

When $\alpha \ne m$ we call a point $r^{\alpha-m}$*-center* of Ω if it gives the minimum value of $V^{(\alpha)}_{\Omega}$ when $\alpha > m$ and the maximum value of $V^{(\alpha)}_{\Omega}$ when $0 < \alpha < m$. When $\alpha = m$ it is meaningless to use $V^{(m)}_{\Omega}$ as it is constantly equal to $\mathrm{Vol}\,(\Omega)$. We call a point an r^0*-center* if it gives the maximum value of the log potential

$$V^{\log}_{\Omega}(x) = \int_{\Omega} \log\frac{1}{|x-y|}\, d\mu(y) = -\int_{\Omega} \log|x-y|\, d\mu(y).$$

As we noticed in the introduction, the center of mass is an r^2-center, and if Ω is convex and $\alpha \ne m$, an $r^{\alpha-m}$-center coincides with the *radial center of order* α, which was introduced in [M] for $0 < \alpha \le 1$ and in [H] for $\alpha \ne 0$.

We remark that the statements of $r^{\alpha-m}$-centers in the case when $\alpha = m$ in this paper can be obtained exactly in the same way as in the case when $0 < \alpha < m$. This is because we only use the estimate on the second derivative in our study, and that of the log potential

$$\frac{\partial^2 V^{\log}_{\Omega}}{\partial {x_j}^2}(x) = \int_{\partial\Omega} |x-y|^{-2}(x_j-y_j) e_j \cdot n \, d\sigma(y)$$

can be considered as the limit of $1/(m-\alpha)$ times the second derivative of $V^{(\alpha)}_{\Omega}$ as α approaches m (see (2.2)).

2.3 Minimal unfolded regions

Let v be a unit vector in S^{m-1} and b be a real number. Put

$$H_{v,b} = \{x \in \mathbb{R}^m \,|\, x \cdot v = b\},\ H^+_{v,b} = \{x \in \mathbb{R}^m \,|\, x \cdot v > b\},\ H^-_{v,b} = \{x \in \mathbb{R}^m \,|\, x \cdot v < b\}.$$

Let $\mathrm{Refl}_{v,b}$ be a reflection of $\mathbb{R}^m$ in $H_{v,b}$. Let Ω be a compact set in $\mathbb{R}^m$. Put $M_v = M_v(\Omega) = \max_{x\in\Omega} x \cdot v$ and

$$u_v = u_v(\Omega) = \inf\Big\{a \,\big|\, a \le M_v,\ \mathrm{Refl}_{v,b}\big(\Omega \cap H^+_{v,b}\big) \subset \Omega \ \ (a \le {}^\forall b \le M_v)\Big\}.$$

The *minimal unfolded region* of Ω is given by

$$Uf(\Omega) = \bigcap_{v\in S^{m-1}} \overline{H^-_{v,u_v}}.$$

It is a non-empty compact convex set and is contained in the convex hull of Ω. It is not necessarily contained in Ω.

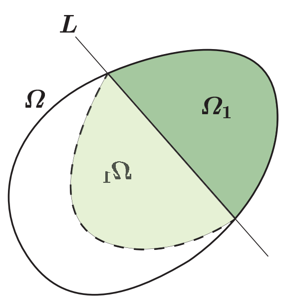

Figure 1: Folding a convex set like origami

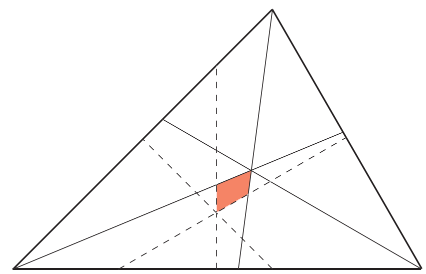

Figure 2: A minimal unfolded region of a non-obtuse triangle. Bold lines are angle bisectors, dotted lines are perpendicular bisectors.

2.4 Existence and uniqueness of $r^{\alpha-m}$-centers

Theorem 2.1 ([O3]) *Let Ω be a body in $\mathbb{R}^m$ with a piecewise C^1 boundary $\partial\Omega$.*

(1) *There exists an $r^{\alpha-m}$-center of Ω for any α.*

(2) *An $r^{\alpha-m}$-center is contained in the minimal unfolded region of Ω for any α.*

(3) *An $r^{\alpha-m}$-center of Ω is unique if $\alpha \ge m+1$.*

(4) *An $r^{\alpha-m}$-center of Ω is unique if $\alpha \le 1$ and Ω is convex.*

The second statement is essentially based on the so-called *moving plane method* in analysis ([GNN]) as the integrands appearing in the formulae of $V_\Omega^{(\alpha)}$ and its derivatives are symmetric. The uniqueness of an $r^{\alpha-m}$-center follows from $\dfrac{\partial^2 V_\Omega^{(\alpha)}}{\partial {x_j}^2} > 0$ when $\alpha \ge m+1$ and the strong concavity $V_\Omega^{(\alpha)}$ on $\mathring{\Omega}$ when $\alpha \le 1$ and Ω is convex.

3 Uniqueness of the centers of $\Omega + \delta B^m$

We conjectured that if Ω is convex then it has only one $r^{\alpha-m}$-center for any α, although it was proved that $V_\Omega^{(\alpha)}$ is not necessarily convex nor concave. In this section we show that the conjeture holds for a δ-parallel bodies $\widetilde{\Omega} + \delta B^m \{x + u \,|\, x \in \widetilde{\Omega},\ u \in B^m\}$ provided that δ is large enough compared with the diameter of $\widetilde{\Omega}$. To be precise, we prove the following theorem:

Theorem 3.1 *For any natural number $m \geq 2$ there is a positive constant $\varphi(m)$ such that for any compact convex set $\widetilde{\Omega}$ in $\mathbb{R}^m$ with piecewise C^1 boundary, if $\delta \geq \varphi(m) \cdot \operatorname{diam}(\widetilde{\Omega})$ then $\widetilde{\Omega} + \delta B^m$ has a unique $r^{\alpha-m}$-center for any α.*

By Theorem 2.1 it is enough to show

$$\frac{\partial^2 V^{(\alpha)}_{\widetilde{\Omega}+\delta B^m}}{\partial {x_j}^2} < 0 \;\; (1 < \alpha < m), \; \frac{\partial^2 V^{\log}_{\widetilde{\Omega}+\delta B^m}}{\partial {x_j}^2} < 0, \; \frac{\partial^2 V^{(\alpha)}_{\widetilde{\Omega}+\delta B^m}}{\partial {x_j}^2} > 0 \;\; (m < \alpha < m+1) \tag{3.1}$$

for any j on the minimal unfolded region of $\widetilde{\Omega} + \delta B^m$.

Lemma 3.2 *Let $\widetilde{\Omega}$ be any compact subset of $\mathbb{R}^m$. The minimal unfolded region of $\widetilde{\Omega} + \delta B^m$ is contained in the convex hull of $\widetilde{\Omega}$ for any $\delta > 0$.*

Proof. Let us use the notation in Subsection 2.3. Let $v \in S^{m-1}$ be any vector and b any real number that satisfies

$$M_v(\widetilde{\Omega}) < b \leq M_v(\widetilde{\Omega} + \delta B^m) = M_v(\widetilde{\Omega}) + \delta.$$

Then for any point Q in $\widetilde{\Omega}$ we have $\operatorname{Refl}_{v,b}\big(B_\delta(Q) \cap H^+_{v,b}\big) \subset B_\delta(Q) \cap H^-_{v,b}$ as the center Q is in $H^-_{v,b}$. As $(\widetilde{\Omega} + \delta B^m) \cap H^+_{v,b} = \bigcup_{Q \in \widetilde{\Omega}} \big(B_\delta(Q) \cap H^+_{v,b}\big)$ we have

$$\begin{aligned} \operatorname{Refl}_{v,b}\big((\widetilde{\Omega} + \delta B^m) \cap H^+_{v,b}\big) &= \bigcup_{Q \in \widetilde{\Omega}} \operatorname{Refl}_{v,b}\big(B_\delta(Q) \cap H^+_{v,b}\big) \\ &\subset \bigcup_{Q \in \widetilde{\Omega}} \big(B_\delta(Q) \cap H^-_{v,b}\big) \\ &\subset (\widetilde{\Omega} + \delta B^m) \cap H^-_{v,b}. \end{aligned}$$

Consequently we have $u_v(\widetilde{\Omega} + \delta B^m) \leq M_v(\widetilde{\Omega})$. It follows that

$$Uf(\widetilde{\Omega} + \delta B^m) = \bigcap_{v \in S^{m-1}} \overline{H^-_{v, u_v(\widetilde{\Omega}+\delta B^m)}} \subset \bigcap_{v \in S^{m-1}} \overline{H^-_{v, M_v(\widetilde{\Omega})}} = \operatorname{conv}(\widetilde{\Omega}).$$

$\square$

Next we proceed to the proof of (3.1) on $\widetilde{\Omega}$.

Definition 3.3 Let m be a natural number with $m \geq 2$ and let $a > 0$. Let L_a denote an oriented line segment in $\mathbb{R}^2$ which starts from $(a, 0)$ and ends at $(0, 1)$. For real numbers α and ξ with $0 \leq \xi < a$ define

$$F(m, \alpha, a, \xi) = \int_{L_a} |x - y|^{\alpha-m-2} (\xi - y_1) \, {y_2}^{m-2} \, dy_2 \,, \tag{3.2}$$

where $x = (\xi, 0)$.

Lemma 3.4 *Suppose $m = 2$ and $1 < \alpha < 3$. For any $a > 0$, if $0 \leq \xi \leq \frac{a}{2}$ then $F(2, \alpha, a, \xi) < 0$.*

Proof. Divide L_a into three parts;

$$L_1 = L_a \cap \{2\xi \leq x_1 \leq a\}, \; L_2 = L_a \cap \{\xi \leq x_1 \leq 2\xi\}, \; L_3 = L_a \cap \{0 \leq x_1 \leq \xi\}.$$

If we put

$$p(t) = \left(\xi + t, \left(1 - \frac{\xi}{a}\right) - \frac{t}{a}\right) \in L_2, \; q(t) = \left(\xi - t, \left(1 - \frac{\xi}{a}\right) + \frac{t}{a}\right) \in L_3 \qquad (0 \leq t \leq \xi)$$

then $|x - p(t)| < |x - q(t)|$ (Figure 3). Hence, as $y_1 = a(1 - y_2)$ on L_a,

$$\int_{L_2 \cup L_3} |x - y|^{\alpha-4} (\xi - y_1) \, dy_2 = \int_0^\xi \left(-|x - p(t)|^{\alpha-4} + |x - q(t)|^{\alpha-4}\right) \cdot t \cdot \frac{dt}{a} < 0.$$

Since $\int_{L_1} |x - y|^{\alpha-4} (\xi - y_1) \, dy_2 < 0$, it completes the proof. $\square$

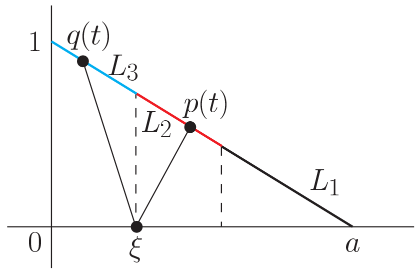

Figure 3:

Lemma 3.5 *Suppose $m \geq 3$ and $1 < \alpha < m+1$. There is a map $\psi_\alpha \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $c > 0$ if $a \geq \psi_\alpha(c)\, c$ then*

$$\int_{-a}^{c} \frac{(t-\frac{1}{a})(t+a)^{m-2}}{\left(\sqrt{t^2+1}\right)^{m+2-\alpha}}\, dt < 0. \tag{3.3}$$

Proof. Let $c > 0$. Assume $a > 2c$. Note that the integrand of (3.3) is positive (or negative) if $t > \frac{1}{a}$ (or respectively $t < \frac{1}{a}$). Therefore we have only to consider the case when $\frac{1}{a} < c$. Observe that

$$\int_{-a}^{c} \frac{(t-\frac{1}{a})(t+a)^{m-2}}{\left(\sqrt{t^2+1}\right)^{m+2-\alpha}}\, dt \leq -\int_{[-2c,-c]\cup[-c,-\frac{1}{a}]} \frac{|t-\frac{1}{a}|(t+a)^{m-2}}{\left(\sqrt{t^2+1}\right)^{m+2-\alpha}}\, dt + \int_{\frac{1}{a}}^{c} \frac{(t-\frac{1}{a})(t+a)^{m-2}}{\left(\sqrt{t^2+1}\right)^{m+2-\alpha}}\, dt, \tag{3.4}$$

where the right hand side can be estimated by

$$\begin{aligned}
\int_{-c}^{-\frac{1}{a}} \frac{|t-\frac{1}{a}|(t+a)^{m-2}}{\left(\sqrt{t^2+1}\right)^{m+2-\alpha}}\, dt &\geq \left(\frac{a-c}{a+c}\right)^{m-2} \int_{\frac{1}{a}}^{c} \frac{(t-\frac{1}{a})(t+a)^{m-2}}{\left(\sqrt{t^2+1}\right)^{m+2-\alpha}}\, dt\,,\\
\int_{-2c}^{-c} \frac{|t-\frac{1}{a}|(t+a)^{m-2}}{\left(\sqrt{t^2+1}\right)^{m+2-\alpha}}\, dt &\geq \left(\frac{a-2c}{a+c}\right)^{m-2} \cdot \frac{1}{\left(\sqrt{4c^2+1}\right)^{m+2-\alpha}} \int_{\frac{1}{a}}^{c} \frac{(t-\frac{1}{a})(t+a)^{m-2}}{\left(\sqrt{t^2+1}\right)^{m+2-\alpha}}\, dt\,.
\end{aligned}$$

As

$$\left(\frac{a-c}{a+c}\right)^{m-2} + \left(\frac{a-2c}{a+c}\right)^{m-2} \frac{1}{\left(\sqrt{4c^2+1}\right)^{m+2-\alpha}} \geq \left(\frac{a-2c}{a+c}\right)^{m-2} \left(1+(4c^2+1)^{-\frac{m+2-\alpha}{2}}\right), \tag{3.5}$$

if we put

$$\psi_\alpha(c) = \frac{2\left(1+(4c^2+1)^{-\frac{m+2-\alpha}{2}}\right)^{\frac{1}{m-2}}+1}{\left(1+(4c^2+1)^{-\frac{m+2-\alpha}{2}}\right)^{\frac{1}{m-2}}-1} = 2 + \frac{3}{\left(1+(4c^2+1)^{-\frac{m+2-\alpha}{2}}\right)^{\frac{1}{m-2}}-1} \tag{3.6}$$

then $a \geq \psi_\alpha(c)c$ is equivalent to

$$\frac{a-2c}{a+c} \geq \left(1+(4c^2+1)^{-\frac{m+2-\alpha}{2}}\right)^{-\frac{1}{m-2}},$$

which implies that the right hand side of (3.4) is negative; thus the proof is completed. Remark that as $\psi_\alpha(c) > 2$, if $a \geq \psi_\alpha(c)\, c$ then a satisfies the assumption $a > 2c$ which appeared at the beginning of the proof. $\square$

Corollary 3.6 *Suppose $m \geq 3$ and $1 < \alpha < m+1$. For any $\xi_0 > 0$ there is $a_0 > 0$ such that if $0 \leq \xi \leq \xi_0$ and $a \geq a_0$ then $F(m,\alpha,a,\xi) < 0$, where $F(m,\alpha,a,\xi)$ is given by* (3.2)*. In fact, we can take*

$$a_0 = \psi_\alpha\left(\frac{2\xi_0^2+1}{\xi_0}\right)\frac{2\xi_0^2+1}{\xi_0},$$

where ψ_α is given by (3.6)*.*

Proof. As $y_1 = a(1-y_2)$ on L_a,

$$F(m,\alpha,a,\xi) = \int_0^1 \left((a(1-y_2)-\xi)^2 + {y_2}^2\right)^{\frac{\alpha-m}{2}-1}(\xi - a(1-y_2))\,{y_2}^{m-2}\,dy_2. \tag{3.7}$$

Put $y_2 - \dfrac{a(a-\xi)}{1+a^2} = \dfrac{a-\xi}{1+a^2}\,t$. Then, as

$$(a(1-y_2)-\xi)^2 + {y_2}^2 = \frac{(a-\xi)^2}{1+a^2}(t^2+1),\ \xi - a(1-y_2) = \frac{a-\xi}{1+a^2}(at-1),\ y_2 = \frac{a-\xi}{1+a^2}(t+a),$$

$F(m,\alpha,a,\xi) < 0$ is equivalent to

$$\int_{-a}^{\frac{a\xi+1}{a-\xi}} \frac{(t-\frac{1}{a})(t+a)^{m-2}}{\left(\sqrt{t^2+1}\right)^{m+2-\alpha}}\,dt < 0.$$

First remark that if $0 < \xi < \xi_0$ then $F(m,\alpha,a,\xi) < F(m,\alpha,a,\xi_0)$ since $\frac{a\xi+1}{a-\xi}$ is an increasing function of ξ $(0 < \xi < a)$ with $\lim_{\xi\to 0}\frac{a\xi+1}{a-\xi} = \frac{1}{a}$ and the integrand is positive when $t > \frac{1}{a}$.

On the other hand, if we put $c(a) = \frac{a\xi_0+1}{a-\xi_0}$, it is a decreasing function of a as $c(a) = \xi_0 + \frac{1+\xi_0^2}{a-\xi_0}$. Put

$$c_0 = c(2\xi_0) = \frac{2\xi_0^2+1}{\xi_0},\ \ a_0 = \psi_\alpha(c_0)c_0 = \psi_\alpha\left(\frac{2\xi_0^2+1}{\xi_0}\right)\frac{2\xi_0^2+1}{\xi_0},$$

where ψ_α is given by (3.6). If $a \geq a_0$ then $c(a) < c(2\xi_0) = c_0$ as $a > 2\xi_0$. Since $c(a) > \frac{1}{a}$ we have

$$\int_{-a}^{c(a)} \frac{(t-\frac{1}{a})(t+a)^{m-2}}{\left(\sqrt{t^2+1}\right)^{m+2-\alpha}}\,dt < \int_{-a}^{c_0} \frac{(t-\frac{1}{a})(t+a)^{m-2}}{\left(\sqrt{t^2+1}\right)^{m+2-\alpha}}\,dt < 0,$$

where the second inequality follows from Lemma 3.5 as $a \geq \psi_0(c_0)c_0$. It implies $F(m,\alpha,a,\xi_0) < 0$, which completes the proof. □

Lemma 3.7 *Suppose $m \geq 2$. There is a function $f\colon (1,m+1) \to \mathbb{R}_+$ with the following property.*

Suppose $\widetilde{\Omega}$ is a convex set of $\mathbb{R}^2_{\geq 0} = \{(x_1,x_2)\,|\,x_2 \geq 0\}$ with a non-empty intersection $\widetilde{\Gamma}_0$ with the x_1-axis. Put $\Omega_\delta = \left(\widetilde{\Omega} + \delta D^2\right) \cap \mathbb{R}^2_{\geq 0}$ and let Γ_δ be the closure of the intersection of $\partial\Omega_\delta$ and the upper half plane (Figure 4)*. Then if $1 < \alpha < m+1$ and if $\delta \geq f(\alpha)\cdot \mathrm{diam}(\widetilde{\Omega})$ then for any point $x = (\xi, 0) \in \widetilde{\Gamma}_0$ we have*

$$\int_{\Gamma_\delta} |x-y|^{\alpha-m-2}(\xi - y_1)\,{y_2}^{m-2}\,dy_2 < 0\,.$$

Proof. Let us write $\Omega = \Omega_\delta$ and $\Gamma = \Gamma_\delta$ in what follows.

Suppose $x \in \widetilde{\Gamma}_0$. If Ω'' is a compact domain that does not contain x then

$$\begin{aligned}&\int_{\partial\Omega''} |x-y|^{\alpha-m-2}(\xi - y_1)\,{y_2}^{m-2}\,dy_2 \\ &= \int_{\Omega''} |x-y|^{\alpha-m-4}\left\{(m+1-\alpha)(y_1-\xi)^2 - {y_2}^2\right\}{y_2}^{m-2}\,dy_1dy_2\end{aligned}$$

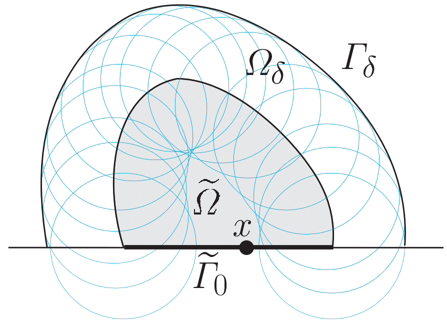

Figure 4: δ-parallel body $\Omega = \Omega_\delta$. Γ_δ is an envelope of circles with radius δ whose centers lie on $\partial\widetilde{\Omega} \cap \mathbb{R}^2_+$

Note that the integrand above is positive if $|y_2| < \sqrt{m+1-\alpha}\,|y_1 - \xi|$ and negative if $|y_2| > \sqrt{m+1-\alpha}\,|y_1 - \xi|$.

Two lines $y_2 = \pm\sqrt{m+1-\alpha}\,(y_1 - \xi)$ intersect Γ in a point each as Ω is convex. Let L be a line through the two intersection points. Remark that the line L does not have any other intersection points with $\partial\Omega$ as Ω is convex. We construct a new domain Ω', which is a rectangle or a triangle,

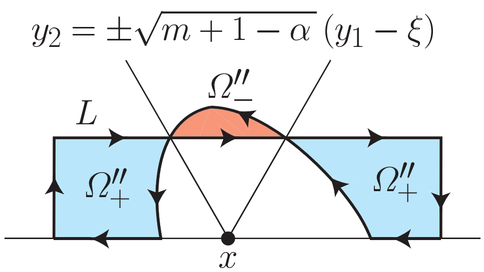

Figure 5: Domain Ω' (the rectangle) when L is parallel to the y_1-axis. The arrows indicate the orientation of $\Omega'' = \Omega - \Omega'$

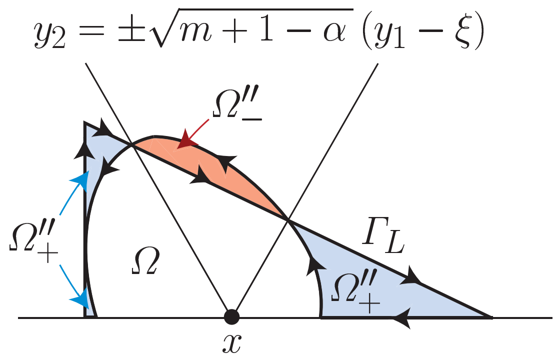

Figure 6: Domain Ω' (the triangle) when L is not parallel to the y_1-axis.

according to whether L is parallel to the y_1-axis or not, bounded by a line segment of L (denoted by Γ_L), a line segment of the y_1-axis, and some vertical line segments as is indicated in figures 6 and 5. When Ω' is a triangle, we take the vertical line segment as close to the point x as possible.

Put $\Omega''_+ = \Omega' \setminus (\Omega' \cap \Omega)$ and $\Omega''_- = \Omega \setminus (\Omega' \cap \Omega)$ (domains in blue (light gray) and red (dark gray) respectively in figures 6 and 5), and $\Omega'' = \Omega - \Omega'$. Note that $\Omega'' = (-\Omega''_+) \cup \Omega''_-$. Then,

$$
\begin{aligned}
&\int_\Gamma |x-y|^{\alpha-m-2}(\xi - y_1)\, {y_2}^{m-2}\, dy_2 = \int_{\partial\Omega} |x-y|^{\alpha-m-2}(\xi - y_1){y_2}^{m-2}\, dy_2 \\
&= \int_{\partial\Omega''} |x-y|^{\alpha-m-2}(\xi - y_1){y_2}^{m-2}\, dy_2 + \int_{\partial\Omega'} |x-y|^{\alpha-m-2}(\xi - y_1)\, {y_2}^{m-2}\, dy_2.
\end{aligned}
\tag{3.8}
$$

As

$$
\mathring{\Omega}''_+ \subset \left\{(y_1, y_2) \,\middle|\, |y_2| < \sqrt{m+1-\alpha}\,|y_1|\right\}, \quad \mathring{\Omega}''_- \subset \left\{(y_1, y_2) \,\middle|\, |y_2| > \sqrt{m+1-\alpha}\,|y_1|\right\},
$$

the first term of the right hand side of (3.8) satisfies

$$
\begin{aligned}
&\int_{\partial\Omega''} |x-y|^{\alpha-m-2}(\xi-y_1)\, {y_2}^{m-2}\, dy_2 \\
&= -\int_{\Omega''_+} |x-y|^{\alpha-m-4}\left\{(m+1-\alpha)(y_1-\xi)^2-{y_2}^2\right\}\, {y_2}^{m-2}\, dy_1 dy_2 \\
&\quad + \int_{\Omega''_-} |x-y|^{\alpha-m-4}\left\{(m+1-\alpha)(y_1-\xi)^2-{y_2}^2\right\}\, {y_2}^{m-2}\, dy_1 dy_2 \\
&< 0.
\end{aligned}
$$

The second term of the right hand side of (3.8) can be estimated as follows. Notice that $\partial\Omega'$ consists of a line segment of L, which we denote by Γ_L, vertical edges, which we denote by Γ_v, and a horizontal edge on the y_1-axis, where the integral vanishes. As the orientation of Γ_v is upward on the right edge and downward on the left edge, we have

$$\int_{\Gamma_v} |x-y|^{\alpha-m-2}(\xi-y_1)\, {y_2}^{m-2}\, dy_2 < 0.$$

Therefore, it remains to show

$$\int_{\Gamma_L} |x-y|^{\alpha-m-2}(\xi-y_1)\, {y_2}^{m-2}\, dy_2 < 0$$

when Γ_L is not parallel to the y_1-axis. It is equivalent to show that $F(m,\alpha,a,\xi)<0$ if a and ξ satisfy some conditions which are derived from the condition for δ.

We may assume without loss of generality that the slope of Γ_L is negative. Put $d=\mathrm{diam}(\widetilde{\Omega})$. Let z (or w) be the intersection point of the y_1-axis and Γ_v (or Γ_L respectively). Let p and q be the intersection points of Γ_L and the lines through x with slopes $\pm\sqrt{m+1-\alpha}$ (Figure 7). Then

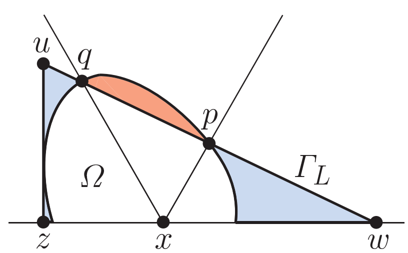

Figure 7:

$$\delta \le |x-p|, |x-q| \le \delta+d.$$

Therefore, the slope of Γ_L is not greater than $\frac{d}{2\delta+d}\sqrt{m+1-\alpha}$, and hence

$$|x-w| \ge \frac{\delta}{\sqrt{m+2-\alpha}} + \frac{\sqrt{m+1-\alpha}}{\sqrt{m+2-\alpha}}\,\delta\,\frac{2\delta+d}{d\sqrt{m+1-\alpha}} = \frac{2\delta(\delta+d)}{d\sqrt{m+2-\alpha}}. \tag{3.9}$$

On the other hand, as we take Γ_v as close to x as poosible, we have

$$|x-z| \le \delta+d. \tag{3.10}$$

(i) Suppose $m=2$. Put $f(\alpha)=\frac{1}{2}\sqrt{4-\alpha}$. Then, if $\delta \ge f(\alpha)d$ then

$$|x-w| \ge \frac{2\delta(\delta+d)}{d\sqrt{4-\alpha}} \ge \delta+d \ge |x-z|.$$

Lemma 3.4 implies $\int_{\Gamma_L} |x-y|^{\alpha-4}(\xi-y_1)\,dy_2 < 0$.

(ii) Suppose $m \ge 3$. Let u be the intersection point of Γ_L and Γ_v. Then $|u-z| \ge \delta\sqrt{\frac{m+1-\alpha}{m+2-\alpha}}$. Put

$$\xi_0 = 2\sqrt{\frac{m+2-\alpha}{m+1-\alpha}}.$$

If we assume $\frac{\delta}{d} \ge 1$ then (3.9) and (3.10) imply

$$\frac{|x-z|}{|u-z|} \le \frac{\delta+d}{\delta}\sqrt{\frac{m+2-\alpha}{m+1-\alpha}} \le \xi_0. \tag{3.11}$$

On the other hand, as the the slope of Γ_L is not greater than $\frac{d}{2\delta+d}\sqrt{m+1-\alpha}$ we have

$$\frac{|w-z|}{|u-z|} \ge \frac{2(\delta+d)}{d\sqrt{m+1-\alpha}} = 2\frac{1+\frac{\delta}{d}}{\sqrt{m+1-\alpha}}. \tag{3.12}$$

Put

$$\begin{aligned} f(\alpha) &= \frac{\sqrt{m+1-\alpha}}{2}\,\psi_\alpha\left(\frac{2\xi_0^2+1}{\xi_0}\right)\frac{2\xi_0^2+1}{\xi_0} - 1 \\ &= \frac{\sqrt{m+1-\alpha}}{2}\left(2 + \frac{3}{\left\{1+\left[4\left(4\sqrt{\frac{m+2-\alpha}{m+1-\alpha}}+\frac{1}{2}\sqrt{\frac{m+1-\alpha}{m+2-\alpha}}\right)^2+1\right]^{-\frac{(m+2-\alpha)}{2}}\right\}^{\frac{1}{m-2}} - 1}\right) \\ &\quad \times \left(4\sqrt{\frac{m+2-\alpha}{m+1-\alpha}}+\frac{1}{2}\sqrt{\frac{m+1-\alpha}{m+2-\alpha}}\right) - 1. \end{aligned} \tag{3.13}$$

Remark that $f(\alpha) \ge 3$ and hence if $\frac{\delta}{d} \ge f(\alpha)$ then the assumption $\frac{\delta}{d} \ge 1$ above is satisfied.

If $\frac{\delta}{d} \ge f(\alpha)$ then (3.12) implies

$$\frac{|w-z|}{|u-z|} \ge \psi_\alpha\left(\frac{2\xi_0^2+1}{\xi_0}\right)\frac{2\xi_0^2+1}{\xi_0}. \tag{3.14}$$

Then by (3.11) and (3.14), Corollary 3.6 implies

$$\int_{\Gamma_L} |x-y|^{\alpha-m-2}(\xi-y_1)\,{y_2}^{m-2}\,dy_2 < 0.$$

□

Theorem 3.8 *Suppose $m \ge 2$ and $1 < \alpha < m+1$. Let $\widetilde{\Omega}$ be a compact convex set in $\mathbb{R}^m$ with a piecewise C^1 boudnary. If $\delta \ge f(\alpha)\cdot \mathrm{diam}(\widetilde{\Omega})$, where $f(\alpha)$ is given in Lemma 3.7, then* (3.1) *holds on $\widetilde{\Omega}$ for any j $(1 \le j \le m)$.*

Proof. Put $\Omega = \widetilde{\Omega} + \delta B^m$ in what follows. Suppose $x \in \widetilde{\Omega}$. By the symmetry, we may assume that $j=1$ and that x is on the x_1-axis. We omit the proof for the case when $\alpha = m$ as it is same as that for the case when $1 < \alpha < m$.

(i) The case when $m=2$. Recall (2.2):

$$\frac{\partial^2 V_\Omega^{(\alpha)}}{\partial {x_1}^2}(x) = (2-\alpha)\int_{\partial\Omega} |x-y|^{\alpha-4}(x_1-y_1)\,dy_2.$$

Divide $\partial\Omega$ into two parts by the x_1-axis, and lemma 3.7 implies the conclusion.

(ii) The case when $m \geq 3$. We use the orthogonal decomposition

$$\mathbb{R}^m = \mathbb{R} \oplus \mathbb{R}^{m-1} = \langle x_1 \rangle \oplus \langle x_2, \dots, x_m \rangle.$$

Suppose the intersection of Ω and the x_1-axis is given by $[x_1^{min}, x_1^{max}]$.

Let S^{m-2} be the unit sphere in $\mathbb{R}^{m-1}$. Suppose $\theta_2, \dots, \theta_{m-1}$ are local coordinates of S^{m-2}. Put $\theta = (\theta_2, \dots, \theta_{m-1})$, and let $\gamma(\theta)$ be the corresponding point on S^{m-2}. Let $\Pi_{\gamma(\theta)}$ be a half 2-plane in $\mathbb{R}^m$ with the axis being the x_1-axis that contains the point $\gamma(\theta)$.

Assume that $\partial\Omega$ can locally be parametrized by

$$\Phi(t,\theta) = (f(t,\theta),\, g(t,\theta)\gamma(\theta)) \in \mathbb{R} \oplus \mathbb{R}^{m-1} \qquad (t_0(\theta) \leq t \leq t_1(\theta))$$

so that the following conditions are satisfied.

- f and g are piecewise C^1-functions with $(f_t)^2 + (g_t)^2 > 0$,
- $f(t_0(\theta),\theta) = x_1^{max}$, $f(t_1(\theta),\theta) = x_1^{min}$,
- $g(t,\theta) \geq 0$, namely $\Phi(t,\theta) \in \Pi_{\gamma(\theta)}$, and $g(t_0(\theta),\theta) = g(t_1(\theta),\theta) = 0$,

Then, if we put $\Gamma_{\gamma(\theta)} = \partial\Omega \cap \Pi_{\gamma(\theta)}$ then $\Gamma_{\gamma(\theta)}$ can be expressed with respect to the x_1-axis and an orthogonal axis in $\Pi_{\gamma(\theta)}$ by (Figure 8)

$$\bar{y}(t,\theta) = (\bar{y}_1, \bar{y}_2) = (f(t,\theta), g(t,\theta)) \qquad (t_0(\theta) \leq t \leq t_1(\theta)).$$

Put

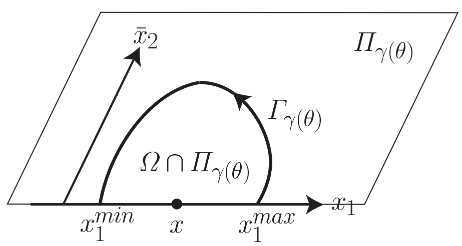

Figure 8:

$$\nu = \frac{\partial\Phi}{\partial t} \times \frac{\partial\Phi}{\partial\theta_2} \times \cdots \frac{\partial\Phi}{\partial\theta_{m-1}},$$

which is a normal vector to $\partial\Omega$. Then ν is an outer normal vector if and only if

$$(\Phi(t,\theta) - p_0) \cdot \nu = \begin{vmatrix} \Phi(t,\theta) - p_0 & \frac{\partial\Phi}{\partial t} & \frac{\partial\Phi}{\partial\theta_2} & \cdots & \frac{\partial\Phi}{\partial\theta_{m-1}} \end{vmatrix} > 0 \tag{3.15}$$

for any point p_0 in $\mathring{\Omega}$ as Ω is convex. When $f_t \neq 0$ we can take p_0 in $\Pi_{\gamma(\theta)}$ so that p_0 has the same x_1-coordinate as $\Phi(t,\theta)$. Then $\Phi(t,\theta) - p_0$ is a positive multiple of $(0, -(\operatorname{sgn} f_t)\gamma(\theta))$. Therefore, if $f_t \neq 0$ then (3.15) is equivalent to

$$\begin{aligned} 0 &< \begin{vmatrix} 0 & f_t & g_{\theta_2} & \cdots & g_{\theta_{m-1}} \\ -(\operatorname{sgn} f_t)\gamma & g_t\gamma & g_{\theta_2}\gamma + g\gamma_{\theta_2} & \cdots & g_{\theta_{m-1}}\gamma + g\gamma_{\theta_{m-1}} \end{vmatrix} \\ &= g^{m-2}|f_t| \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \gamma & \gamma_{\theta_2} & \cdots & \gamma_{\theta_{m-1}} \end{vmatrix} \\ &= g^{m-2}|f_t| \begin{vmatrix} \gamma & \gamma_{\theta_2} & \cdots & \gamma_{\theta_{m-1}} \end{vmatrix}. \end{aligned}$$

Assume $\theta_2, \ldots, \theta_{m-1}$ are positive local coordinates of S^{m-2}, i.e. $\left|\gamma \ \gamma_{\theta_2} \ \cdots \ \gamma_{\theta_{m-1}}\right| > 0$. Then ν is an outer normal vector to $\partial\Omega$. This holds even when $f_t = 0$ because in this case ν is outer normal if and only if $(\operatorname{sgn} g_t) e_1 \cdot \nu > 0$, which follows from (3.16) below.

On the other hand,

$$\begin{aligned} e_1 \cdot \nu &= \begin{vmatrix} 1 & f_t & g_{\theta_2} & \cdots & g_{\theta_{m-1}} \\ 0 & g_t\gamma & g_{\theta_2}\gamma + g\gamma_{\theta_2} & \cdots & g_{\theta_{m-1}}\gamma + g\gamma_{\theta_{m-1}} \end{vmatrix} \\ &= g^{m-2} g_t \left|\gamma \ \gamma_{\theta_2} \ \cdots \ \gamma_{\theta_{m-1}}\right|. \end{aligned} \tag{3.16}$$

Since

$$\begin{aligned} dS^{m-2} &= \left|\gamma \ \gamma_{\theta_2} \ \cdots \ \gamma_{\theta_{m-1}}\right| \, d\theta_2 \cdots d\theta_{m-1}, \\ d\sigma &= |\nu| \, dt \, d\theta_2 \cdots d\theta_{m-1}, \\ n &= \nu / |\nu|, \end{aligned}$$

we have

$$\begin{aligned} e_1 \cdot n \ d\sigma &= g^{m-2} g_t \left|\gamma \ \gamma_{\theta_2} \ \cdots \ \gamma_{\theta_{m-1}}\right| \, dt \, d\theta_2 \cdots d\theta_{m-1} \\ &= g^{m-2} g_t \, dt \, dS^{m-2}. \end{aligned}$$

Therefore, (2.2) implies that

$$\begin{aligned} \frac{\partial^2 V_\Omega^{(\alpha)}}{\partial {x_1}^2}(x) &= (m-\alpha) \int_{\partial\Omega} |x-y|^{\alpha-m-2} (x_1 - y_1) \, e_1 \cdot n \, d\sigma(y) \\ &= (m-\alpha) \int_{S^{m-2}} \left(\int_{t_0(\theta)}^{t_1(\theta)} |x-\bar{y}|^{\alpha-m-2} (x_1 - \bar{y}_1) g^{m-2} g_t \, dt \right) dS^{m-2}. \end{aligned}$$

By lemma 3.7

$$\int_{t_0(\theta)}^{t_1(\theta)} |x-\bar{y}|^{\alpha-m-2} (x_1 - \bar{y}_1) g^{m-2} g_t \, dt = \int_{\Gamma_{\gamma(\theta)}} |x-\bar{y}|^{\alpha-m-2} (x_1 - \bar{y}_1) \, {\bar{y}_2}^{m-2} \, d\bar{y}_2 < 0$$

for each point $\gamma(\theta)$ in S^{m-2}, which completes the proof. □

Corollary 3.9 *Suppose $m \geq 2$ and $1 < \alpha \leq m+1$. For any compact convex set $\widetilde{\Omega}$ in $\mathbb{R}^m$ with a piecewise C^1 boundary, if $\delta \geq f(\alpha) \cdot \operatorname{diam}(\widetilde{\Omega})$, where $f(\alpha)$ is given in Lemma 3.7, then $\widetilde{\Omega} + \delta B^m$ has a unique $r^{\alpha-m}$-center.*

When $m = 2$ we have $\sup_{1<\alpha<3} f(\alpha) = \sqrt{3}$, so if we put $\varphi(2) = \sqrt{3}$ we completes the proof of Theorem 3.1 for the case when $m = 2$.

When $m \geq 3$, unfortunately we have $\sup_{1<\alpha<m+1} f(\alpha) = +\infty$ as $\lim\limits_{\alpha \nearrow m+1} f(\alpha) = +\infty$.

Lemma 3.10 *Suppose $m \geq 3$. For any $b > 0$ there is $\alpha_0 = \alpha_0(b)$ with $m < \alpha_0 < m+1$ such that for any compact convex set $\widetilde{\Omega}$ in $\mathbb{R}^m$ with a piecewise C^1 boundary, if $\delta \geq b \cdot \operatorname{diam}(\widetilde{\Omega})$ then $\widetilde{\Omega} + \delta B^m$ has a unique $r^{\alpha-m}$-center if $\alpha_0 \leq \alpha < m+1$.*

Proof. Suppose $\widetilde{\Omega}$ has diameter d and $x \in \widetilde{\Omega}$. Let $C_j(\alpha)$ be the cone with vertex x given by

$$C_j(\alpha) = \left\{ y \, \middle| \, -(m+1-\alpha)(x_j - y_j)^2 + \sum_{i \neq j} (x_i - y_i)^2 \leq 0 \right\}.$$

The radial function of $\widetilde{\Omega}+\delta B^m$ with respect to x defined by $\rho(v)=\sup\{t\geq 0 \,|\, x+tv\in\widetilde{\Omega}+\delta B^m\}$ ($v\in S^{m-1}$) satisfies $\delta\leq\rho(v)\leq\delta+d$ for any v. Therefore

$$\begin{aligned}
&\frac{1}{\alpha-m}\cdot\frac{\partial^2 V^{(\alpha)}_{\widetilde{\Omega}+\delta B^m}}{\partial {x_j}^2}(x)=\int_{\widetilde{\Omega}+\delta B^m}|x-y|^{\alpha-m-4}\left(-(m+1-\alpha)(x_j-y_j)^2+\sum_{i\neq j}(x_i-y_i)^2\right)d\mu(y)\\
&\geq\int_{B^m_{\delta+d}(x)\cap C_j(\alpha)}|x-y|^{\alpha-m-4}\left(-(m+1-\alpha)(x_j-y_j)^2+\sum_{i\neq j}(x_i-y_i)^2\right)d\mu(y)\\
&+\int_{B^m_{\delta}(x)\cap C_j(\alpha)^c}|x-y|^{\alpha-m-4}\left(-(m+1-\alpha)(x_j-y_j)^2+\sum_{i\neq j}(x_i-y_i)^2\right)d\mu(y).
\end{aligned}\tag{3.17}$$

Define $g\colon(m,m+1)\times\mathbb{R}_+\to\mathbb{R}$ by

$$g(\alpha,\beta)=\int_{X_{\alpha,\beta}}\frac{-(m+1-\alpha){y_1}^2+\sum_{i>1}(x_i-y_i)^2}{|y|^{m+4-\alpha}}\,d\mu(y),$$

where

$$X_{\alpha,\beta}=B^m\cup\left(B^m_{1+\frac{1}{\beta}}\cap C_1(\alpha)\right).$$

Remark that $g(\alpha,\beta)$ is an increasing function of β. Fix $b>0$. As $g(\alpha,b)$ is continuous with respect to α and $g(m+1,b)>0$, there is $\alpha_0\in(m,m+1)$ such that if $\alpha_0\leq\alpha<m+1$ then $g(\alpha,b)>0$, which completes the proof as the right hand side of (3.17) is proportional to $g(\alpha,b)$. □

Suppose $m\geq 3$. Put

$$\varphi(m)=\max\left\{10,\sup_{1<\alpha\leq\alpha_0(10)}f(\alpha)\right\}=\max\left\{10,\max_{1\leq\alpha\leq\alpha_0(10)}f(\alpha)\right\},$$

where $f(\alpha)$ is given by (3.13) and α_0 is given in Lemma 3.10. Then, Theorem 3.8 and Lemma 3.10 implies Theorem 3.1 for the case when $m\geq 3$.

Remark 3.11 In [O3], using the same renormalization process of defining energy functionals of knots ([O1], [O2]), we renormalized $V^{(\alpha)}_{\Omega}(x)$ so that it is well-defined for $\alpha\leq 0$ and $x\in\mathring{\Omega}$. The $r^{\alpha-m}$-center of Ω for $\alpha\leq 0$ can be defined in a similar way: it is a point in $\mathring{\Omega}$ where $V^{(\alpha)}_{\Omega}\big|_{\mathring{\Omega}}$ attains the maximum value. We can show $\dfrac{\partial^2 V^{(\alpha)}_{\Omega}}{\partial {x_j}^2}<0$ on $\mathring{\Omega}$ for any j if $\alpha\leq 1$ and Ω is convex by a similar way as in Lemma 3.7 and Theorem 3.8. This gives an alternative proof of the uniqueness of the $r^{\alpha-m}$-center of Ω when $\alpha\leq 1$ and Ω is convex.

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